Intermediation, The Stock Market and Intergenerational Transfers

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General Equilibrium: Problems, Prospects and Alternatives

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Roughly speaking there are two classes of general equilibrium models which explicitly recognize that the future is open ended: the first, simplest and most idealized, views agents as being infinitely lived—they are all permanently on stage together. With a complete set of markets such an economy leads to an efficient outcome: in essence all contingencies can be traded out at an initial date and the future unfolds in a preordained way thereafter. The second, more realistic framework, recognizes that agents are finitely lived—agents in any generation are on stage for only a transient interval of time and are replaced by the agents of the next generation. In this latter model, markets face more serious problems for co-ordinating agents decisions over time. In this paper we use the simplest overlapping generations (OLG) model with production to study how capital markets co-ordinate decisions of consumers and the investment decisions of firms, their joint decisions leading to a path of capital accumulation for the economy.

The distilled essence of our message is best understood by first looking at the simplest model of an overlapping generations exchange economy, namely the canonical model of Allais (1947) and Samuelson (1958), as analyzed and interpreted in the elegant paper of Gale (1973). The first principle that we draw from this preliminary model is that economies can be classified into one of two types depending on the direction (sign) of the transfers between generations at the Golden Rule (positive if from young to old, negative if from old to young) and that the long-run dynamic behavior of an economy can be predicted once the sign of these transfers is known. The second principle is that financial markets lead to long-run efficiency for economies with negative transfers, but do not lead to long-run efficiency for economies with positive transfers, i.e. financial markets can successfully transfer income backwards, but not forwards.

We then consider the simplest model of an overlapping generations production economy, namely the classic model of Diamond (1965), which has become one of the basic workhorses of modern macroeconomics. Many current policy debates – for example on the nature and role of the social security system – draw heavily on the insights of this model, and it is here that our message comes with a warning. The Diamond model, and the host of literature that ensued (see for example Blanchard-Fischer (1989)) assumes that borrowing and lending can only occur between agents of the same generation – in short, that the bond market is intragenerational. Once the smallest dose of intergenerational intermediation is introduced, the equilibrium of the model changes radically for all economies characterized by negative transfers at the efficient steady state. This can be seen by analyzing a generalized version of the Diamond model in which infinitely-lived intermediaries permit intergenerational transfers to take place on the bond market. The resulting model generates equilibria whose qualitative behavior parallels that of the OLG exchange economies. The equilibria of economies with positive transfers at the Golden Rule, (called economies with *overaccumulation*), converge to the Diamond steady state, which is inefficient: at such a steady state too much of the society's current output is devoted to investment rather than consumption. Thus the enlarged set of market opportunities offered by intergenerational intermediation does not lead to long-run efficiency for economies with overaccumulation. By contrast, for economies with negative transfers at the Golden Rule (called economies with *underaccumulation*), the equilibrium converges to the Golden Rule. In this case the enlarged financial opportunities offered by the financial intermediaries lead to greater long-run efficiency than that obtained with the more restrictive lending structure of the Diamond model. If, as is often argued in the literature (e.g. Abel et al. (1989)), underaccumulation is the most realistic case, and if, as we would argue, there is a rich set of financial intermediaries which permit intergenerational transfers, then many of the important policy recommendations which are based on the Diamond model, can change in significant ways. For example, while a Diamond model predicts that a pay-as-you-go social security system lowers the long-run level of capital, in the model with intergenerational transfers, even though a pay-as-you-go system slows down the process of capital accumulation, it has no impact on its long-run behavior.

Finally we argue that the bond market is not the only financial market through which intergenerational transfers can take place. Provided "firms" are modeled in a more realistic way, it can be shown that the canonical market for the transfer of ownership of firms, the *stock market*, provides a natural mechanism for the intergenerational transfer of funds. The greater realism consists in recognizing that capital once installed in a firm can no longer be unbolted and made equivalent to so many units of current output, in short that capital is a sunk cost. In such a setting, we show that the financial value of a firm is typically less than its replacement cost (i.e. has a Tobin's q less than one) and that this discount on the value of the firm relative to its replacement cost in essence provides a mechanism for the transfer of funds from the old generation that owns the firm to the young generation which purchases it. Thus the framework of this paper throws a new light on the role of the stock market as an instrument for transferring ownership of firms across generations.

A. OLG Exchange Economy

Consider an overlapping generations exchange economy with one good, in which agents live for 2 periods and population grows at the rate n. Let N_t denote the number of young born at date t, then $N_t = (1+n)N_{t-1}$ where N_{t-1} is the number of old at date t. Each agent has a lifetime endowment $e = (e_0, e_1) \in \mathbb{R}^2_+$, representing income when young and old, and a preference ordering represented by a utility function $u(c_0, c_1)$, where (c_0, c_1) is the agent's lifetime consumption stream: thus all agents are identical, except for their date of birth. Let $\mathcal{E}(u, e, n)$ denote the associated exchange economy in which each agent has the utility function u, the endowment stream e, and the population grows at the rate n. In the analysis that follows we will restrict attention to a subset of preference orderings which leads to a simple structure for the dynamics of the equilibrium model. This subset is most conveniently defined by drawing on the standard framework of microeconomic theory which tells us how an agent's demand responds to changes in prices and income: the two "goods" are consumption when young and old (c_0, c_1), their prices are (p_0, p_1), and the agent's income (wealth) under these prices is denoted by $m = p_0 e_0 + p_1 e_1$.

Assumption U. The utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$ satisfies

(i) u is strictly quasi-concave, smooth, increasing, satisfies $u'_0(c) \to \infty$ if $c_0 \to 0^+$ and $u'_1(c) \to \infty$ if $c_1 \to 0^+$ (where $u'_\ell = \dfrac{\partial u}{\partial c_\ell}$, $\ell = 0, 1$)

(ii) the demand function $c(p, m) = \arg\max\{u(c) \,|\, pc = m\}$ with $c(p, m) = (c_0(p, m), c_1(p, m))$ satisfies

(α) $\dfrac{\partial c_0}{\partial m}(p, m) > 0$, $\dfrac{\partial c_1}{\partial m}(p, m) > 0$, for all $p \gg 0$, $m > 0$

(β) $\dfrac{\partial c_0}{\partial p_0}(p, pe) < 0$, $\dfrac{\partial c_1}{\partial p_0}(p, pe) > 0$ for all $p \gg 0$ and $e \in \mathbb{R}^2_+$ such that $pe > 0$

(ii)(α) requires that consumption at each period of life be a normal good; (ii)(β) requires that when the income and substitution effects are of opposite signs, the substitution effect dominates. In the sequential model with borrowing and lending at the interest rate r, we will see that $(p_0, p_1) = ((1+r), 1)$: thus an increase in p_0 is equivalent to an increase in the rate of interest r and (ii)(β) implies an increase in the rate of interest decreases date 0 consumption and increases date 1 consumption. Assumption U is satisfied by Cobb-Douglas utility functions and CES utility functions with elasticity of substitution greater than 1.

We assume that the story starts at an initial date, called date $t = 0$. Let $c_t = (c_{0,t}, c_{1,t+1})$ denote the (lifetime) consumption stream of a representative agent of the generation born at date t. An *allocation* $\boldsymbol{c} = (c_{1,0}, (c_t)_{t\geq 0})$ is a consumption stream $(c_t)_{t\geq 0})$ for all generations born from date $t = 0$ into the indefinite future, and consumption $c_{1,0}$ for the representative agent of the old generation at $t = 0$. Note that a consequence of imposing a starting date on the model ($t = 0$) is that the model cuts into the life story of agents of the very first generation in "midlife", specifying only what their consumption $c_{1,0}$ is in their old age and leaving completely unspecified what their consumption was when they were young. This asymmetric treatment of the first generation is an unavoidable consequence of the overlapping structure of the generations and the desire to start the model at some date in the finite past which we call date $t = 0$ — and not at $t = -\infty$ as the model would really like. Some of the tricky and unintuitive properties of the OLG model have their origin in this asymmetric treatment of the first generation. An allocation $\boldsymbol{c} = (c_{1,0}, (c_t)_{t\geq 0})$ is *feasible* if for all $t \geq 0$

$$N_t c_{0,t} + N_{t-1} c_{1,t} = N_t e_0 + N_{t-1} e_1 \iff (1+n)(c_{0,t} - e_0) + (c_{1,t} - e_1) = 0 \tag{1}$$

namely if no more consumption is distributed to the young and old generations at date t than is available through their combined date t endowments. An allocation is *Pareto optimal* if there is no other feasible allocation $\tilde{c} = (\tilde{c}_{1,0}, (\tilde{c}_t)_{t\geq 0})$ such that $\tilde{c}_{1,0} \geq (c_{1,0})$ and $u(\tilde{c}_t) \geq u(c_t)$ for all $t \geq 0$, with at least one strict inequality.

The simplest kind of allocation is one in which all generations have the same lifetime consumption: more precisely, we say that the allocation $\boldsymbol{c} = (c_{1,0}, (c_t)_{t\geq 0})$ is a *steady state* if there exists a lifetime consumption stream (c_0, c_1) such that $c_{1,0} = c_1,\ c_t = (c_0, c_1)$ for all $t \geq 0$. For a steady state (c_0, c_1), the feasibility condition (1) becomes

$$(1+n)c_0 + c_1 = (1+n)e_0 + e_1 \tag{2}$$

A steady state (c_0^*, c_1^*) is said to be *efficient* if it satisfies (2) and there is no other feasible steady state which is preferred by the representative agent of any generation. An efficient steady state (c_0^*, c_1^*) is a solution to the problem

$$\max\{u(c) \,|\, (1+n)c_0 + c_1 = (1+n)e_0 + e_1, \quad c \in \mathbb{R}^2_+\}$$

and is characterized by the first-order condition

$$\frac{u_0'(c_0^*, c_1^*)}{u_1'(c_0^*, c_1^*)} = 1 + n \tag{3}$$

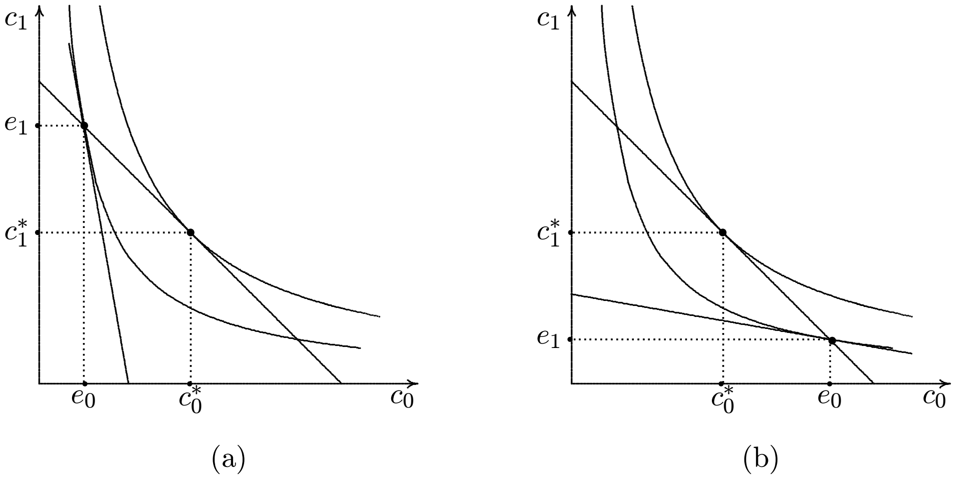

Figure 1: The Golden Rule (c_0^*, c_1^*): for the economy in (a) the endowment (e_0, e_1) is such that the transfers at (c_0^*, c_1^*) are negative; for the economy in (b), (e_0, e_1) is such that the transfers are positive.

The geometric representation of the efficient steady state is the same as the geometric representation of the optimal choice of consumption of an agent with initial endowment (e_0, e_1) maximizing utility under the budget constraint (2) (see Figure 1). Under assumption U there is a unique efficient steady state (c_0^*, c_1^*) which is called the *Golden Rule (steady state).*

It follows from Gale's (1973) analysis that there is an interesting way of classifying the economies $\mathcal{E}(u, e, n)$ depending on the behavior of the savings of the young generation at the Golden Rule. This classification partitions economies into two types, with the dynamical behavior of an economy within each type having the same qualitative properties.

Definition. An exchange economy $\mathcal{E}(u, e, n)$ is said to have *negative (positive) transfers at the Golden Rule* if $s* = e_0 - c_0^* < 0$ (> 0).

To abbreviate, we will refer to these economies as *negative transfer economies* and *positive transfer economies* respectively. The hairline case where $s^* = 0$ is ignored since it is exceptional. For given preferences u and demographic structure n (the indifference curve and the slope of the line in Figure 1) an economy has negative (positive) transfers if the endowment in old age is sufficiently large (small) relative to that in youth — Figure 1(a) and (b) respectively. Because classical economists typically considered economies in which income is increasing, while Samuelson in his original paper (1958) studied the case where endowments in old age are zero, Gale (1973) refers to the cases of negative and positive transfers as the "classical" and "Samuelson" cases respectively.

The rate of impatience $r(c_0, c_1)$ of an agent at a consumption stream $(c_0, c_1) \in \mathbb{R}^*_+$ is defined by

$$1 + r(c_0, c_1) = \frac{u'_0(c_0, c_1)}{u'_1(c_0, c_1)}$$

Let $r^* = r(c^*_0, c^*_1) = n$ be the rate of impatience at the *Golden rule* and let $r_A = r(e_0, e_1)$ be the rate of impatience at the *autarchic* steady state (initial endowment). From the geometry of the indifference curves in Figure 1, note the following useful property: *$r_A > r^*$ in an economy with negative transfers and $r_A < r^*$ when the transfers are positive.*

Note that the notion of an efficient steady state is different from the notion of a Pareto optimal allocation, since the comparison is restricted to other steady states. The difference comes from the awkward presence of the representative old agent at date 0. For example, the autarchic steady state where agents consume their initial endowment is typically an inefficient steady state. However in the case of negative transfers, it is a Pareto optimal allocation: as seen from Figure 1(a) the allocations which are preferred to the initial endowment by the representative agent involve consuming more at date 0. But then feasibility implies that the representative old agent at date 0 consumes less, so that a Pareto improvement is impossible.

Market Structure and Intermediation. So far we have analyzed the feasible allocations of the economy $\mathcal{E}(u, e, n)$: now it is time is to study the allocations that can arise when the agents resort to decentralized trading on markets. Our objective is to study the consequence of introducing a financial market structure in which agents can borrow and lend so as to achieve a lifetime consumption stream $c_t = (c_{0,t}, c_{1,t+1})$ which is preferred to their initial endowment stream $e = (e_0, e_1)$. It is clear that the contemporary young and old generations at date t cannot *directly* enter into such loan contracts, since the old will be dead next period. To make intergenerational loans possible we assume that there is an infinite-lived financial intermediary[1]: if young agents want to borrow, they borrow from the intermediary which collects their payments (principal and interests) when they are old, and uses the proceeds to lend to the new young generation. To ensure that the model can start off with funds available to lend to the young, we apply a "symmetry" condition to the old generation, assuming that in their youth ($t = -1$) they borrowed in a way similar to the current ($t = 0$) young, so that they are willing to reimburse in their old age. A similar reasoning applies when the young want to lend: they give their savings to the intermediary which uses them to pay back the old who saved in the previous period. In this latter case the intermediary can be replaced by an asset like "money" which can be carried over from one period to the next. The young buy

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[1] More accurately we assume that there is a large number of these intermediaries so that perfect competition prevails and the intermediaries make zero profit on their constant-returns activity.

it from the old and sell it to the new young generation when they have themselves become old.

Suppose therefore that a young agent born at date t can borrow or lend with an interest rate r_{t+1} to be paid at date $t+1$. The agent's sequential budget set is given by[2]

$$\mathbb{B}(r_{t+1}) = \left\{ c_t \in \mathbb{R}^2_+ \;\middle|\; \begin{array}{rcll} c_{0,t} & = & e_0 - s & \\ c_{1,t+1} & = & e_1 + (1+r_{t+1})s, & s \in \mathbb{R} \end{array} \right\} \tag{4}$$

Let $c(r_{t+1})$ denote the solution to the agent's problem of maximizing utility over the budget set $\mathbb{B}(r_{t+1})$

$$c(r_{t+1}) = \arg\max\{u(c) |\, c \in \mathbb{B}(r_{t+1})\}$$

$c(r_{t+1})$ is summarized by the optimal savings function $s(r_{t+1})$ satisfying $(c_0(r_{t+1}), c_1(r_{t+1})) = (e_0 - s(r_{t+1}),\; e_1 + (1+r_{t+1})s(r_{t+1}))$. Note that $s(r_A) = 0$ and $s(r^*) = e_0 - c_0^*$ (see Figure 1).

Since at date t there are $(1+n)$ young agents for each old agent, the equilibrium on the market for loans (the budget constraint of the intermediary) imposes the conditions

$$(1+n)s(r_{t+1}) = (1+r_t)s(r_t) \quad \forall t \geq 0 \tag{E}$$

When the model begins at date 0, the interest rate r_0, or equivalently the consumption $c_{1,0}$ of the old agents at date 0, must be given as initial condition.

Definition. An *intermediary equilibrium* for the OLG exchange economy $\mathcal{E}(u, n, e)$, with initial condition r_0, is a pair $((\boldsymbol{c}, \boldsymbol{s}),\ \boldsymbol{r}) = \left(c_{1,0}, (c_t, s_t)_{t=0}^{\infty}, (r_{t+1})_{t=0}^{\infty}\right)$ consisting of a sequence of consumption-saving decisions $(\boldsymbol{c}, \boldsymbol{s})$ and a sequence of (correctly anticipated) interest rates $\boldsymbol{r}$ such that $c_{1,0} = s(r_0)(1+r_0)$, $c_t = c(r_{t+1})$, $s_t = s(r_{t+1})$, and (E) holds for all $t \geq 0$.

Equation (E) is the difference equation which, together with the initial condition r_0, determines the equilibrium path of interest rates. Its steady state solutions are $r^* = n$ and r_A, where the latter is the only interest rate r such that $s(r) = 0$. Define the functions $\mathcal{D}$ and $\mathcal{S}$ by

$$\mathcal{D}(r) = (1+n)s(r), \qquad \mathcal{S}(r) = (1+r)s(r)$$

At date t, the current interest rate is r_{t+1} and $\mathcal{D}(r_{t+1})$ is the "demand" function of the young — a demand for loans if $\mathcal{D}(r_{t+1}) < 0$, and a demand for savings opportunities if $\mathcal{D}(r_{t+1}) > 0$ — while $\mathcal{S}(r_t)$ is the "supply" of funds forthcoming from the old, which is fixed at date t since it depends on the previous period interest rate r_t. The equilibrium conditions (E) can be written as

$$\mathcal{D}(r_{t+1}) = \mathcal{S}(r_t), \quad \forall\, t \geq 0$$

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[2] Note that by a standard argument, which involves multiplying the date t constraint by $(1+r_{t+1})$ and adding it to the constraint for date $t+1$, the budget set $\mathbb{B}(r_{t+1})$ is equivalent to the budget set $B(r_{t+1}) = \{c_t \in \mathbb{R}^2_+ | (1+r_{t+1})c_{0,t} + c_{1,t+1} = (1+r_{t+1})e_0 + e_1\}$.

An intermediary equilibrium is thus a sequence of interest rate $(r_t)_{t \geq 0}$ such that at each date t, the interest rate r_{t+1} adjusts so as to equate the current demand $\mathcal{D}(r_{t+1})$ to the fixed supply of funds $\mathcal{S}(r_t)$ determined by the previous period interest rate r_t.

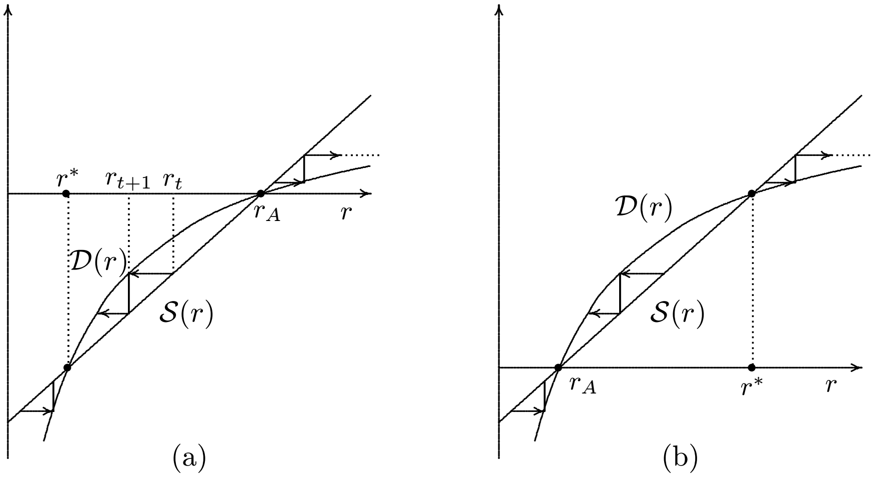


Figure 2: The equilibrium interest rate r_t changes to r_{t+1} to equate demand $\mathcal{D}(r_{t+1})$ to supply $\mathcal{S}(r_t)$ of loans. (a) for negative transfer economy r_t converges to r^*; (b) for positive transfer economy r_t converges to r_A.

Under Assumption U, $\mathcal{D}(\cdot)$ and $\mathcal{S}(\cdot)$ are monotone increasing functions: their graphs are shown in Figure 2(a) for the case $r^* < r_A$, and in Figure 2(b) for the case $r_A < r^*$. Since $s(r) < 0$ whenever $r < r_A$ and $s(r) > 0$ whenever $r > r_A$, the curve $\mathcal{D}$ must lie below the curve $\mathcal{S}$ when $r < r^* < r_A$ in case (a) or $r < r_A < r^*$ in case (b), and when $r > \max(r^*, r_A)$. Whenever at any interest rate r, $\mathcal{D}(r) > \mathcal{S}(r)$, the interest rate must be reduced to clear the loan market, while if $\mathcal{D}(r) < \mathcal{S}(r)$ the interest rate must be increased. As a result for every economy (with $r^* \neq r_A$) *the low interest rate steady state is stable, and the high interest rate steady state in unstable.* For a negative transfer economy ($r^* < r_A$), the equilibrium converges to the Golden Rule for all initial conditions such that $r_0 < r_A$, the equilibrium stays in the autarchic steady state if $r_0 = r_A$, and if $r_0 > r_A$ then the interest rate increases to infinity and there is no equilibrium starting from r_0. *Thus in an economy with negative transfers the presence of a financial intermediary, by permitting transfers from the old to the young, makes it possible for the economy to reach a long-run efficient steady state, for all viable initial conditions, except* $r_0 = r_A$.

For a positive transfer economy ($r_A < r^*$), the equilibrium converges to the autarchy steady state for all initial conditions $r_0 < r^*$; the equilibrium stays at the Golden rule steady state if $r_0 = r^*$, and if $r_0 > r^*$ there is no equilibrium. *Unless the interest rate happens to be exactly right* *initially* ($r_0 = r^*$)*, the economy is driven back to autarchy—an allocation which, in addition of being an inefficient steady state, is not even Pareto optimal.*

What is very striking in this model is how *the demographic factor — the population growth rate* n *— determines the outcome on the financial market*, namely the behavior of equilibrium interest rates. To understand why it plays such a pervasive role, note first that by the definition of r_A and the monotonicity of each young agent's saving function, if $r < r_A$ young agents when faced with the interest rate r will want to borrow, while if $r > r_A$ they will want to lend (save). Now consider a negative transfer economy with an interest rate $r_t < n < r_A$. Then old agents have borrowed in their youth (at $t-1$) and are reimbursing at date t. However since $r_t < n$ they do not collectively as a cohort reimburse enough to permit each young agent to borrow as much as they themselves had borrowed at date $t-1$: the sheer weight of the new cohort is too great to permit the loan market to be cleared at the same interest rate. Thus each young agent must be induced to reduce his borrowing, and the only way to do this is to raise the interest rate, $r_{t+1} > r_t$. On the other hand if the interest rate r_t exceeds the rate of growth of the population ($n < r_t < r_A$), then the cohort of old agents reimburse in such a way that each young agent at date t must be induced to borrow more than the old agents had borrowed at the previous date, and the only way to do this is to lower the interest rate $r_{t+1} < r_t$. Thus for a negative transfer economy ($n < r_A$) it is the sheer weight of the cohort of young agents pouring out on the loan market (relative to the existing cohort of old agents) which drives the interest rate to equality with the rate of population growth n. The same reasoning applies in a positive transfer economy ($r_A < n$), except that in this case for an interest rate r_t close to n, young agents save instead of borrowing. Thus if $r_A < r_t < n$, the old agents must be reimbursed with the savings of the young. However, since $r_t < n$, the cohort of young agents is too large and young agents must be induced to save less than the previous generation and the only way this can be achieved is by lowering the interest rate $r_{t+1} < r_t$. If $r_t > n$, the young agents must be induced to save more than the previous generation and the interest rate has to increase, $r_{t+1} > r_t$. After a while however it is no longer possible to induce young agents to save more to reimburse the old generation, since the income of the young is bounded: adjustments in the rate of interest do not permit demand to be matched to supply and the system breaks down. Thus if $r_t > n$ in case (b) (or by the same reasoning if $r_t > r_A$ in case (a)) there is no equilibrium.

An important message that emerges from the OLG exchange model is that *competitive financial markets can achieve long-run efficiency when the Golden Rule requires transfers from the old to the young (negative transfers), but cannot prevent the economy from reverting to autarchy in the case where agents save at the Golden Rule (positive transfers).* For economies in this latter category, it seems that only direct transfers from young to old agents — akin to a pay-as-you-go social security system — can restore long-run efficiency.

B. OLG Production Economy

In this section we shall show that the insight obtained from the OLG model of an exchange economy regarding the role of intergenerational intermediation carries over in a natural and interesting way to a production economy. The canonical model of an OLG production economy was introduced in the classic paper of Diamond (1965). While the exact market structure underlying the model can be expressed in a number of equivalent ways, the basic idea is that certain agents of the young generation (who can be thought of as entrepreneurs) buy the capital stock of existing firms from the old generation entrepreneurs, and then decide on the new investment they want to make in their firms. The funds required to finance both the purchase of firms and the new investment is obtained by loans from fellow members of the young cohort: thus borrowing by firms from consumers is assumed to be *intragenerational*. In the resulting equilibrium, capital accumulates to an inefficient steady state, called the Diamond steady state.

We shall show that when intermediaries are introduced which make *intergenerational intermediation* possible, the dynamic properties of the equilibrium change in a radical way, For economies in which the Golden Rule level of capital requires substantial borrowing to purchase and maintain the efficient capital stock, relative to the willingness of the young to save (the analogue of a negative transfer economy in the production setting), in equilibrium capital accumulates to the long-run efficient steady state. Thus long-run efficiency is restored by removing the constraint on borrowing imposed on the entrepreneurs of the young generation, and the increased borrowing power made possible by the intermediaries re-establishes the long-run efficiency of markets. In the opposite case where the borrowing required to finance the efficient capital stock is small relative to the willingness of the young to save (the analogue of a positive transfer economy), in equilibrium capital accumulates to the inefficient steady state: for these economies, just as in the exchange setting, efficiency can only be restored by resorting to government intervention in the form of a social security tax which directly transfers from the young to the old. There is thus a complete parallel between the role of intergenerational intermediation and social security for the attainment of long-run efficiency in the exchange and production economy settings. In the production model the Diamond steady state plays the role of autarchy in the exchange model—autarchy corresponding to the absence of intergenerational transfers. Whether or not intermediation plays an important role in facilitating the appropriate capital accumulation in the economy depends on the relation between two fundamentally different characteristics of the economy, the willingness of the agents to save and the technological and demographic factors determining the Golden Rule.

We show that the analysis can be pushed a step further. We introduce an additional element of realism into the modeling of firms by recognizing that, since the resources invested in a firm typically have to be adapted in a way that is *firm specific* to make them function smoothly and efficiently, the expenses involved in the purchase of such resources cannot readily be recovered on a resale market. Thus, to simplify, we consider the theoretical limit where capital, once installed in a given firm, has no value in a general market for second-hand capital. In order to maintain the value of its invested capital a firm must retain its identity as an income generating entity in the economy: the natural market for transmitting firms across generations then becomes a market for ownership rights to the future stream of income that it will generate, namely a stock market for ownership shares in the firm. Now, as in Tobin's q theory (Tobin(1969)), the *replacement cost* of a firm and its *stock market value* become two distinct valuations. We exploit this property to show that under certain conditions, the intergenerational intermediation required in an economy with "negative transfers" can be achieved through the stock market which transfers firms between the old and the new generation at a cost which is lower than the replacement cost of the firm. Thus we obtain a new insight into the role of a stock market as an instrument leading to long-run efficiency by permitting cheaper transfer of firms across generations. To show these results we begin by recalling Diamond's model of an OLG production economy.

Diamond's Model

The demographic structure is the same as in the exchange model: agents live for two periods and at date t there are N_t young and N_{t-1} old agents with $N_t = (1+n)N_{t-1}$. There are three goods, a consumption good, a capital good and labor. Each agent has an endowment of one unit of labor when young and zero units when old. All agents have the same preference orderings over lifetime consumption represented by a utility function $u(c_0, c_1)$, with no disutility in the provision of labor. We will continue to assume that Assumption U is satisfied.

In the way Diamond originally presented the model, and in subsequent accounts given in the literature (for example Blanchard-Fischer (1989)), there is no fully articulated micro-market structure which describes the complete functioning of the economy. Furthermore there are a number of different, albeit equivalent, market structures within which the model can be embedded: we shall choose the one which provides the most natural and convenient reference point for presenting our generalized version of the Diamond model.

We consider the following market structure. At each date t there are four markets: a market for current output, a labor market, a second-hand capital market and a bond market. At each date there is a market on which the *current output* of firms is sold: this output can be used either as a consumption good or as a new capital good to add to the existing stock of capital. Since this is a real (as opposed to a monetary) model, the price of this good is normalized to be 1 in each period: this simply means that it acts as the numeraire in terms of which all valuations are expressed. In each period there is also a *labor market* on which the (homogeneous) services of the labor supplied by the young generation are sold to the firms, at the current wage rate w_t.

Since capital is durable, a market is needed for transferring the ownership of this capital between generations. Since previously installed capital is assumed to be a perfect substitute for new capital (much to the horror of vintage capital theorists), the price of *second-hand capital* is the same as the price of new capital, provided there is positive investment (i.e. young agents do not want to disinvest): we will restrict the analysis to this case.

The fourth market is the *bond market*, which permits young agents to save for their retirement and firms to finance their capital investment: the interest rate on a loan from date t to $t+1$ is r_{t+1}.

Some of the young agents born at date t are entrepreneurs: without loss of generality, the firm set up by each entrepreneur is a *sole proprietorship* and there are J_t such firms (we will see shortly that the number of these firms and the exact identity of the entrepreneurs does not matter, given that the assumption of constant returns to scale will be invoked). To create firm j the entrepreneur finances the capital input by borrowing from fellow members of his cohort on the bond market. This capital input, which we call K^j_{t+1}, because it becomes operational in the following period $t+1$, is purchased in part on the second-hand capital goods market and in part on the market for current output (the new capital goods component). When this capital K^j_{t+1} is combined at date $t+1$ with an amount L^j_{t+1} of labor services, it gives rise to an output $F^j_{t+1}(K^j_{t+1}, L^j_{t+1})$. The key simplifying assumption on which the Diamond and all growth models rest is that the production function F^j_{t+1} is time homogeneous, exhibits constant returns to scale and is the same for all firms.

Each firm, like the entrepreneur that owns it, lives for two periods: it is financed and created in period t and generates its output $F(K^j_{t+1}, L^j_{t+1})$ in the next period $t+1$. The entrepreneur must pay the wage bill $(w_{t+1}L^j_{t+1})$, reimburse the principal and interest on the loan $((1+r_{t+1})K^j_{t+1})$, and then sells the remaining used capital $((1-\beta)K^j_{t+1})$ on the second-hand capital goods market, where β is the depreciation rate of capital. The entrepreneur's profit is thus

$$\pi^j_{t+1} = F(K^j_{t+1}, L^j_{t+1}) - w_{t+1}L^j_{t+1} - (1 + r_{t+1})K^j_{t+1} + (1 - \beta)K^j_{t+1}$$

Assuming that the prices (w_{t+1}, r_{t+1}) are such that profit maximization by firms has a positive solution, the first-order condition for the optimal choice of labor L^j_{t+1} is given by

$$F'_L\left(K^j_{t+1}, L^j_{t+1}\right) = F'_L\left(\frac{K^j_{t+1}}{L^j_{t+1}}, 1\right) = w_{t+1} \iff f\left(k^j_{t+1}\right) - k^j_{t+1}f'\left(k^j_{t+1}\right) = w_{t+1} \tag{5}$$

where $f(k^j) = F(k^j, 1)$ is the production function per unit of labor, and $k^j = K^j/L^j$. In the same way the first-order condition for the optimal choice of K^j_{t+1} is given by

$$F'_K\left(\frac{K^j_{t+1}}{L^j_{t+1}}, 1\right) = \beta + r_{t+1} \iff f'(k^j_{t+1}) \;=\; \beta + r_{t+1} \tag{6}$$

The first-order conditions (5) and (6) show that profit maximization for firm j only determines the capital-labor $k^j_{t+1} = K^j_{t+1}/L^j_{t+1}$ for firm j at date $t+1$, and that this ratio is the same for all firms. Note also that Euler's theorem for homogeneous functions of degree 1 implies that at the optimal capital-labor choice, firm j's profit is zero ($\pi^j_{t+1} = 0$).

In his youth, each agent must make a lifetime consumption plan: a wage w_t is received for the unit of labor supplied to one of the firms when young. A part of this income must be saved on the bond market to provide income for the agent in old age. Since entrepreneurs earn no profit (in equilibrium) from the ownership of their firms, the consumption-savings decision of every young agent is the same and involves choosing $(c_t, s_t) = (c_{0,t}, c_{1,t+1}, s_t)$ which solves the problem

$$\max \left\{ u(c_t) \;\middle|\; \begin{array}{lcl} c_{0,t} & = & w_t \;-\; s_t, \quad s_t \in \mathbb{R} \\ c_{1,t+1} & = & s_t(1 + r_{t+1}) \end{array} \right\} \tag{7}$$

As in the exchange case, the solution is entirely characterized by the function $s : \mathbb{R}^2_+ \longrightarrow \mathbb{R}$ where $s(r_{t+1}, w_t)$ denotes the amount saved by a representative young agent faced with the interest rate r_{t+1} and the wage rate w_t.

At each date t there are four markets that need to clear: those for (current) output, labor, second-hand capital and bonds with prices $(1, w_t, 1, r_{t+1})$ respectively. The wage rate w_t must clear the labor market: since K^j_t has been inherited from the previous period, L^j_t is chosen so that the capital-labor ratio K^j_t/L^j_t satisfies (5) (at date t). Since all firms have the same capital-labor ratio, so that $k^j_t = \sum_{j=1}^{J_t} K^j_t / \sum_{j=1}^{J_t} L^j_t$ for all j, in order that the labor market clear ($\sum_{j=1}^{J_t} L^j_t = N_t$), the aggregate capital stock of all firms at date t, $K_t = \sum_{j=1}^{J_t} K_t^j$, and the wage rate w_t must be linked by the conditions

$$f(k_t) - k_t f'(k_t) = w_t \tag{8}$$

$$k_t = \frac{K_t}{N_t} \tag{9}$$

Equilibrium on the bond market at date t requires that the savings of young consumers equals the borrowing by the entrepreneurs (of the same cohort), namely that

$$N_t s(r_{t+1}, w_t) = \sum_{j=1}^{J_t} K_{t+1}^j (r_{t+1}, w_{t+1})$$

All entrepreneurs choose K_{t+1}^j, anticipating the same capital-labor ratio k_{t+1} at date $t+1$ satisfying

$$f'(k_{t+1}) = \beta + r_{t+1} \tag{10}$$

so that $\sum_{j=1}^{J_t} K_{t+1}^j (r_{t+1}, w_{t+1}) = \sum_{j=1}^{J_t} L_{t+1}^j k_{t+1}$.

Given that their anticipations are correct $\left(\sum_{j=1}^{J_t} L_{t+1}^j = N_{t+1}\right)$

$$\sum_{j=1}^{J_t} K_{t+1}^j (r_{t+1}, w_{t+1}) = N_{t+1} k_{t+1}$$

so that equilibrium on the bond market will occur if

$$N_t s_t(w_t, r_{t+1}) = N_{t+1} k_{t+1} \iff s_t(w_t, r_{t+1}) = (1+n) k_{t+1} \tag{11}$$

Equilibrium on the second-hand capital market only requires that (net) investment defined by $I_t \equiv K_{t+1} - (1-\beta) K_t$ be non-negative. Entrepreneurs are indifferent between second-hand capital or new capital goods, so that we can assume that they begin by buying second-hand capital before drawing on the current output of firms to create “new” capital goods. In the analysis that follows we restrict attention to trajectories for which $I_t > 0$. The relation defining net investment when expressed in per-capita terms becomes

$$(1+n) k_{t+1} = (1-\beta) k_t + i_t$$

where $i_t = I_t / N_t$. Note that by Walras' Law, equilibrium on the labor and bond markets implies that the market for the (current) output of firms clears. The zero profit condition for all firms implies that

$$\begin{aligned} Y_t &= w_t N_t + (1+r_t) K_t - (1-\beta) K_t \\ &= N_t (c_{0,t} + s_t) + N_{t-1} c_{1,t} - K_{t+1} + I_t \\ &= N_t c_{0,t} \; + \; N_{t-1} c_{1,t} + I_t \end{aligned}$$

so that supply equals demand for current output at date t.

Equations (8) - (11) together with an appropriate initial condition define an equilibrium of the model. Note that only aggregate variables enter into these equations, so that the model is in essence a *macro model*. As we have seen it is compatible with a micro-structure for consumers and firms but the exact structure of the firms (the number of firms, who the owners are and how capital and labor are allocated among firms) does not affect the equilibrium consumption streams of agents. In this sense the equilibrium of the model is invariant to the exact microstructure of firms, and for this reason is usually studied directly at the aggregate level on the production side.

The model begins at date 0 with a capital stock K_0 inherited from the previous period $t = -1$: the condition that K_0 was chosen optimally at date -1 requires that

$$f'\left(\frac{K_0}{N_0}\right) = \beta + r_0 \tag{12}$$

and the per-capita savings of the date 0 old generation must have been such that $\frac{N_0}{1+n}s_{-1} = K_0$. Thus

$$c_{1,0} = (1+n)k_0(1+r_0) \tag{13}$$

where r_0 is given by (12).

Definition. A *Diamond equilibrium* of the OLG production economy $\mathcal{E}(u, F, n)$ is a sequence $(\bar{\boldsymbol{c}}, \bar{\boldsymbol{w}}, \bar{\boldsymbol{k}}, \bar{\boldsymbol{r}}) = \left(\bar{c}_{1,0}, (\bar{c}_t, \bar{w}_t, \bar{k}_{t+1}, \bar{r}_{t+1})_{t\geq 0}\right)$ such that

(i) $\bar{c}_{1,0}$ is given by (12) and (13)

(ii) $\bar{c}_t = (\bar{c}_{0,t}, \bar{c}_{1,t+1})$ is given by (7) with $s_t = s(w_t, r_{t+1})$ for all $t \geq 0$

(iii) $(\bar{w}_t, \bar{k}_{t+1}, \bar{r}_{t+1})$ satisfy (8) - (11) for all $t \geq 0$.

Given (8) and (10) the trajectory of the per-capita aggregate capital stock $(k_t)_{t\geq 0}$ entirely determines the equilibrium. Inserting (8) and (10) into (11) gives the first-order difference equation

$$s(r(k_{t+1}), w(k_t)) = (1+n)k_{t+1}, \quad \forall t \geq 0 \tag{D}$$

which together with the initial condition k_0 determines the equilibrium path of capital. Assumption U implies the following properties:

- $0 < s'_w(r, w) < 1$

- $s'_r(r, w) \geq 0$

An additional assumption is needed to ensure that the equilibrium sequence $(k_t)_{t\geq 0}$ has a simple (monotonic) behavior

Assumption S. The function $k \to s(r(k), w(k))/k$ is decreasing with

$$\lim_{k\to 0^+} \frac{s(r(k), w(k))}{k} > 1+n, \quad \lim_{k\to\infty} \frac{s(r(k), w(k))}{k} = 0$$

Although this assumption is a joint assumption on preferences and technology, it can be decomposed into separate assumptions on the consumption and the production sides. For example, it holds if

- u is homothetic and satisfies Assumption U
- f is such that $w(k)/k$ is a decreasing function of k with $\lim_{k\to 0^+} \dfrac{w(k)}{k} = \infty$ and $\lim_{k\to\infty} \dfrac{w(k)}{k} = 0$

These conditions are satisfied if both u and F are CES with elasticity of substitution greater than or equal to 1. The proof of the following result can be found in Magill-Quinzii (1999): it simply makes explicit the assumptions required to obtain Diamond's original result. The geometric interpretation is shown in figure 3.

Proposition. *Under Assumptions (U, S) the Diamond equilibrium trajectory $(k_t)_{t\geq 0}$ has the following properties:*

(i) *there exists an increasing function ϕ such that (D) is equivalent to*

$$k_{t+1} = \phi(k_t)$$

(ii) *ϕ has a unique fixed point k_D such that*

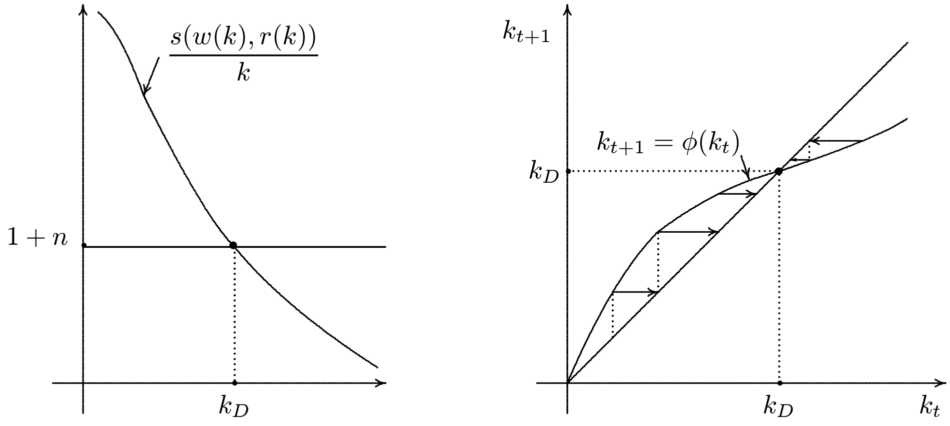
$$k_D = \phi(k_D) \iff s(r(k_D), w(k_D)) = (1+n)k_D$$

(iii) *If $k_0 > 0$, the sequence $(k_t)_{t\geq 0}$ determined by (D) converges monotonically to k_D.*

Is k_D an efficient steady state? To answer this question let us consider the feasible steady states of the economy, i.e. the feasible allocations $(\boldsymbol{c}, \boldsymbol{k})$ such that $c_t = (c_0, c_1)$ for all $t \geq 0$, $c_{1,0} = c_1$, $k_t = k$ for all $t \geq 0$, for some $(c_0, c_1, k) \in \mathbb{R}^3_+$. Feasibility implies

$$N_t c_0 + N_{t-1} c_1 + N_t i = N_t f(k)$$

.



(a)

(b)

Figure 3: (a) the function $k \to s(r(k), w(k))/k$; (b) convergence to the Diamond steady state.

where i is the (per-capita) investment which permits the (per-capita) capital to be constant i.e. $N_t(1-\beta)k + N_t i = N_{t+1}k \iff i = (n+\beta)k$. The feasibility condition reduces to

$$c_0 + \frac{c_1}{1+n} = f(k) - (n+\beta)k$$

and the steady state which maximizes the permanent per-capita consumption, called the *Golden Rule*, is characterized by

$$f'(k^*) = n + \beta$$

Since the Golden rule is entirely determined by the technological (f, β) and demographic (n) factors while the Diamond steady state depends in addition on agents' preferences, typically $k_D \neq k^*$. Note that $r(k^*) = n$ is the interest rate which should prevail in order to induce firms to choose the capital-labor ratio k^*. A typical economy $\mathcal{E}(u, F, n)$ is such that either

(i) $s(r(k^*), w(k^*)) < (1+n)k^*$ or (ii) $s(r(k^*), w(k^*)) > (1+n)k^*$

economies for which the equality holds being exceptional. Let

$$Z(k_t, k_{t+1}) = s(r(k_{t+1}), w(k_t)) - (1+n)k_{t+1}$$

denote the transfers from the young generation to the old that would be needed if the young generation were to carry over to date $t+1$ the per-capita capital stock k_{t+1} when they receive a wage $w(k_t)$ and the interest rate $r(k_{t+1})$ justifies the choice of k_{t+1}. The economies $\mathcal{E}(u, F, n)$ can thus be classified by the sign of the transfers at the Golden Rule

Definition. A production economy $\mathcal{E}(u, F, n)$ is said to have *negative (positive) transfers at the Golden Rule* if $Z(k^*, k^*) < 0$ (> 0).

Assumption S implies that economies with negative transfers are such that $k_D < k^*$ (see Figure 3(a)) and $r_D > n$ (since f is concave). The savings of the young agents at k^* are not sufficient to sustain the Golden Rule. Thus in order that at the Diamond steady state the investment of the firms can be financed out of the savings of the young the interest rate has to be higher, to induce both a lower investment from firms and (weakly) more savings from the young agents. This case is referred to in the literature as the case of *underaccumulation*. Economies with positive transfers are such that $k_D > k^*$ and $r_D < r^*$: the savings of the young agents at k^* are too high for the capital stock needed at the Golden Rule. In order to absorb the savings, at the Diamond steady state the interest rate is lower, both to induce firms to invest more and to discourage agents from saving. This is the case of *overaccumulation*.

Intermediation in a Production Economy

There is thus an interesting parallel between the OLG exchange and production models, the Diamond steady state playing the role in production economies of the autarchy steady state in exchange economies. This suggests that the inefficiency of the Diamond steady state comes from the constraint that the capital stock carried by the generation born at date t to date $t+1$ is entirely financed out of their savings. Let us show that when this constraint is removed by assuming the existence of an *infinitely-lived intermediary* which permits intergenerational transfers, then the Diamond steady state is unlikely to occur in economies with negative transfers (underaccumulation), while it remains the stable steady state in economies with positive transfers (overaccumulation). One way of seeing the difference between the two models is that in Diamond's model borrowing and lending goes through short-lived intermediaries (banks) which are created by the agents of the young generation and die with this generation in the subsequent period, while in the model that we now consider the intermediaries are infinitely-lived corporations transmitted from one generation to the next. While short-lived intermediaries restrict borrowing and lending within a given generation, long-lived intermediaries permit intergenerational transfers.

Consider the same market structure as in the Diamond model (markets for current output, labor, second-hand capital, borrowing and lending) but in which the borrowing and lending goes through an infinitely-lived intermediary; young agents, as consumers, give their savings to the intermediary and as entrepreneurs borrow from the intermediary; the old agents take out their savings and as entrepreneurs reimburse what they had borrowed for financing the investment of their firms. The output, capital market and labor markets function exactly as before: at date t consumers and firms face the prices $(1, w_t, 1, r_{t+1})$ and make their optimizing decision in the same way. The choice of a consumer of generation t is described by the savings function $s(r_{t+1}, w_t)$, all firms choose the same capital labor ratio, and the equilibrium equations (8), (9), and (10) have to hold. The only change is that the market clearing equation (11) has to be replaced by the budget constraint of the intermediary

$$N_t s_t - K_{t+1} - N_{t-1}(1+r_t)s_{t-1} + (1+r_t)K_t = 0$$

or, in per capita terms

$$(1+n)\Big(s(r_{t+1}, w_t) - (1+n)k_{t+1}\Big) = (1+r_t)\Big(s(r_t, w_{t-1}) - (1+n)k_t\Big) \tag{14}$$

When the economy starts at date 0 all choices inherited from the past must be given as initial conditions, in particular the initial capital stock k_0, which as in the previous section determines the interest rate r_0 which prevailed at date -1. But (k_0, r_0) no longer determines $c_{1,0}$: one needs to know how much the old have saved in their youth, which depends on k_{-1} through $s(r_0, w_{-1})$. Thus the initial conditions must be (k_{-1}, k_0). Then

$$r_0 = f'(k_0) - \beta, \quad w_{-1} = f((k_{-1}) - k_{-1}f'(k_{-1}), \quad c_{1,0} = (1+r_0)s(r_0, w_{-1}) \tag{15}$$

Definition. A *intermediary equilibrium* of the OLG production economy $\mathcal{E}(u, F, n)$ is a sequence $(\bar{\boldsymbol{c}},, \bar{\boldsymbol{w}}, \bar{\boldsymbol{k}}, \bar{\boldsymbol{r}}) = \Big(\bar{c}_{1,0}, (\bar{c}_t, \bar{w}_t, \bar{k}_{t+1}, \bar{r}_{t+1})_{t\geq 0}\Big)$ such that

(i) $\bar{c}_{1,0}$ is given by (15)

(ii) $\bar{c}_t = (\bar{c}_{0,t}, \bar{c}_{1,t+1})$ is given by (7) with $s_t = s(w_t, r_{t+1})$ for all $t \geq 0$

(iii) $(\bar{w}_t, \bar{k}_{t+1}, \bar{r}_{t+1})$ satisfy (8),(9), (10), and (14) for all $t \geq 0$.

An intermediary equilibrium, like a Diamond equilibrium, is determined by the sequence of per-capita capital stocks $(k_t)_{t\geq 0}$. This sequence is determined by the second-order difference equation

$$(1+n)\Big(s(r(k_{t+1}), w(k_t)) - (1+n)k_{t+1}\Big) = (1+r(k_t))\Big(s(r(k_t), w(k_{t-1})) - (1+n)k_t\Big)$$

or

$$(1+n)Z(k_{t+1}, k_t) = (1+r(k_t))Z(k_t, k_{t-1}), \quad \forall t \geq 0 \tag{E$'$}$$

with initial condition (k_{-1}, k_0). Equation (E$'$) is the second-order version of the first-order difference equation (E) which determines the interest rate in the exchange model. It expresses the equality of the demand for funds by the young to the supply of funds by the old . The steady state equilibria of (E$'$) are k^*, defined by $r(k^*) = n$, and k_D defined by $Z(k_D, k_D) = 0$.

Under Assumption U, the difference equation (E$'$) can be written as

$$k_{t+1} = \psi(k_t, k_{t-1}), \quad \forall t \geq 0$$

where ψ is a differentiable function for $(k_t, k_{t-1}) \gg 0$. If we define the new variable $v_t = k_{t-1}$ then this second-order difference equation can be written as the following equivalent first-order system in (k_t, v_t)

$$\begin{array}{rcl} v_{t+1} & = & k_t \\ k_{t+1} & = & \psi(k_t, v_t) \end{array} \quad \forall t \geq 0 \tag{16}$$

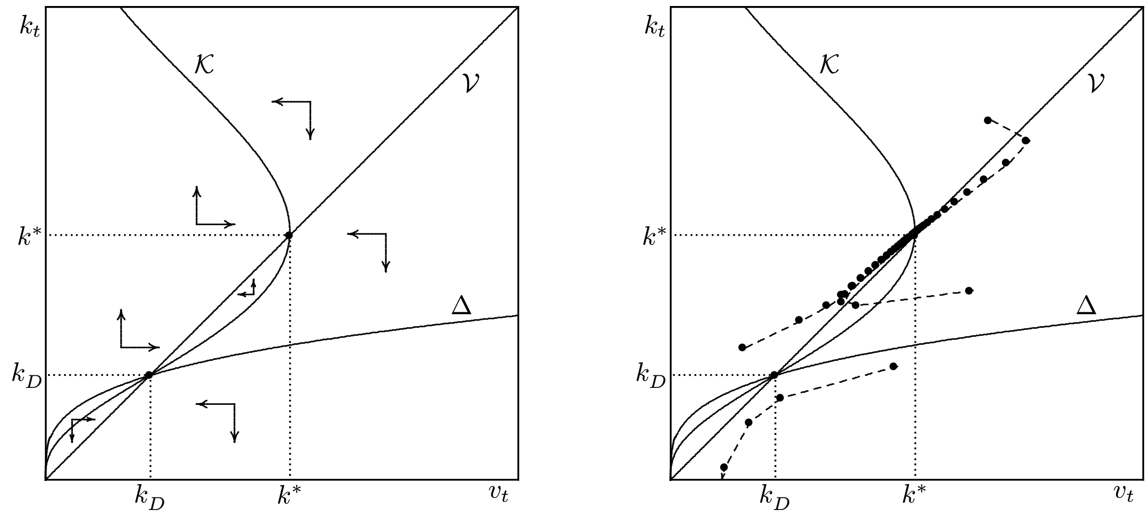
with initial condition $(k_0, v_0) \gg 0$. The simplest way of getting a feel for the qualitative properties of the trajectories defined by (16) is to examine the associated phase portrait shown in Figures 4(a) and 5(a) for economies with underaccumulation and overaccumulation respectively. To construct the phase diagram, we need to construct the curve $\mathcal{V}$ consisting of the points (k_t, v_t) in the plane such that $v_{t+1} = v_t$, and the curve $\mathcal{K}$ of points (k_t, v_t) such that $k_{t+1} = k_t$. Depending on which side of the $\mathcal{K}$ and $\mathcal{V}$ curves a point (k_t, v_t) of a trajectory is located, k_{t+1} will be greater or smaller than k_t and v_{t+1} will be greater or smaller than v_t. As indicated in Figures 4(a) and 5(b), $\mathcal{V}$ is just the diagonal ($v_{t+1} = v_t$ is equivalent to $k_t = v_t$), while the curve $\mathcal{K}$ is bell-shaped: there is a value $\bar{k} > \max(k_D, k^*)$ such that if v_t (i.e. k_{t-1}) is less than $\bar{k}$, then there are two values of k_t such that $k_{t+1} = k_t$, and if v_t is greater than $\bar{k}$, there is no value of k_t such that $k_{t+1} = k_t$.

Just as we found that in the exchange model the dynamics of equilibrium was such that the smallest interest rate steady state was stable, so here in the model with production the phase diagram suggests that *the steady state with the highest level of capital (and hence the lowest interest rate) is the stable steady state.*

The third curve Δ in Figures 4(a) and 5(a) is the curve on which the sequence of points (k_t, v_t) of any Diamond equilibrium must lie

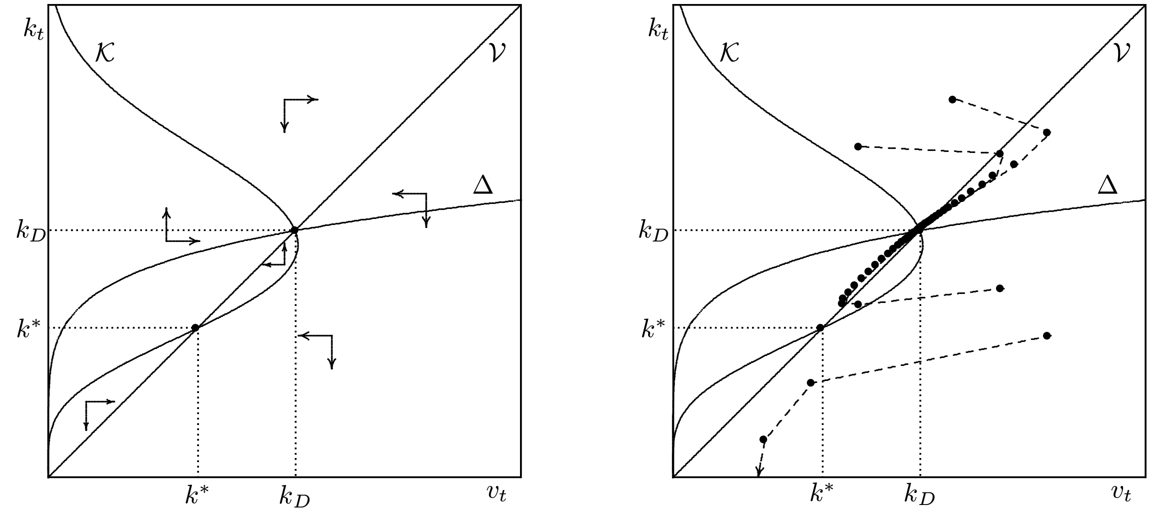
$$\Delta = \{(k_t, v_t) \in \mathbb{R}^2_+ \mid Z(k_t, v_t) = 0\}$$

namely the pairs (k_t, v_t) satisfying (D). Note that if $(k_0, v_0) \in \Delta$ then for all $t \geq 0, Z(k_t, v_t) = 0$ and the trajectory is just the Diamond equilibrium converging to (k_D, k_D). If (k_0, v_0) lies above (below) Δ, then equation (E$'$) implies that $Z(k_t, v_t) < 0$ (resp > 0) for all $t \geq 0$: along the equilibrium trajectory the total capital on hand at any date t exceeds (is less than) the savings of the young.



(a) (b)

Figure 4: Intermediary equilibria converging to the Golden Rule in the case of underaccumulation.



(a) (b)

Figure 5: Intermediary equilibria converging to the Diamond steady state in the case of overaccumulation.

As is well-known, for a difference equation system, the phase diagram is not sufficient to establish the dynamical properties of the trajectories, even though it suggests the qualitative behavior of the trajectories. A global result for the system (E$'$) can be obtained for economies with overaccumulation (see Tirole (1985) and Magill-Quinzii (1999)). For economies with underaccumulation a global result seems more difficult to obtain—although the trajectories starting above the curve Δ converge in all examples with Cobb Douglas and CES functions we have tried (see Figures 4(b) and 5(b)). In both cases a local analysis can be carried out around the two positive steady states (k_D, k_D) and (k^*, k^*), and the nature of the two eigenvalues can be established: one root inside and one outside the unit circle for (k_D, k_D) (i.e. locally saddle point), and both inside the unit circle for (k^*, k^*) (i.e. locally stable) for an economy with underaccumulation; both roots inside the unit circle for (k_D, k_D), and one inside, the other outside for (k^*, k^*) for an economy with overaccumulation. Thus the following proposition can be established.

Proposition: *Consider OLG production economies $\mathcal{E}(u, F, n)$ satisfying Assumption (U, S). An intermediary equilibrium of $\mathcal{E}$ is characterized by (E') and has the following properties.*

(i) *For an economy with positive transfers (overaccumulation) the Diamond steady state is locally stable and the Golden Rule steady state is saddlepoint stable. All trajectories with initial conditions satisfying $Z(k_0, k_{-1}) < 0$ converge to the Diamond steady state. If $Z(k_0, k_{-1}) > 0$ either the trajectory converges to the Diamond steady state, or there is no equilibrium, or the trajectory converges to the Golden Rule if (k_0, k_{-1}) lies on the one-dimensional stable manifold leading to k^*.*

(ii) *If in addition $kf'(k)$ is an increasing function of k then for an economy with negative transfers (underaccumulation) the Golden Rule steady state is locally stable and the Diamond steady state is saddlepoint stable. A trajectory converges to the Diamond steady state if and only if $Z(k_0, k_{-1}) = 0$. The trajectories such that $Z(k_0, k_{-1}) < 0$ converge to the Golden Rule or oscillate around it. If $Z(k_0, k_{-1}) > 0$ there is no equilibrium.*

An OLG model embeds demography into an equilibrium model in an essential way, the key demographic parameter being the rate of population growth n. As we have seen, for both a production and an exchange economy this parameter determines the Golden Rule interest rate. In Section A we showed how, in an exchange economy, the force of population growth also determines the long-run dynamics of the economy: the adjustment of the interest rate to cope with the constant inflow of new agents essentially explains why the "low interest rate" equilibrium is stable and the "high interest rate equilibrium" is unstable. The inflow of new agents onto the capital markets has a similar effect in a production economy, although the way in which it affects the equilibrium is more involved since the interest rate influences both the investment decisions of firms and the savings decisions of the consumers, and the savings decisions are in addition influenced by the endogenously determined wage rate.

However it is still possible to get the intuition for the long-run properties of the equilibrium by considering some simple cases. For example, consider an economy with underaccumulation and suppose that $k_D < k_{t-1} < k_t < k^*$. Then $Z(k_t, k_{t-1}) < 0$ and its absolute value is the excess per-capita reimbursement of firms over the amount needed to reimburse old agents. Since $k_t < k^*$, $r_t > n$, and the new young generation must be induced to "overspend" on investment more per capita than the previous generation $\Big(|Z(k_{t+1}, k_t)| > |Z(k_t, k_{t-1})|\Big)$ despite the fact that they have higher income ($< k_{t-1} < k_t \Longrightarrow w_{t-1} < w_t$). Thus, to clear the capital market, r_{t+1} must be lower than r_t, implying that k_{t+1} is greater than k_t. If, on the other hand, $k_D < k^* < k_t < k_{t-1}$ then $Z(k_t, k_{t-1})$ is still negative, but since $r_t < n$ the young generation must be induced to overspend less per-capita than the previous generation $\Big(|Z(k_{t+1}, k_t)| < |Z(k_t, k_{t-1})|\Big)$ even though they have a lower income. Thus the interest rate must rise, so that the capital stock decreases. Thus, once again, for a negative transfer economy ($Z(k^*, k^*) < 0$) it is the weight of the cohort of young agents which flows onto the loan market which drives the interest rate into equality with the rate of population growth, inducing thereby a process of capital accumulation which converges to the Golden Rule k^*.

Now consider an economy with overaccumulation and suppose that $k_t < k_{t-1} < k^* < k_D$. Then $Z(k_t, k_{t-1}) > 0$ and the firms reimburse less than is needed to pay back the principal and interest on the savings of the old agents: the difference must be made up by the excess savings of the young. Since $k_t < k^*$, r_t is greater than n and the young agents must be induced to have higher excess savings per capita than the previous generation, even though they have a smaller income: thus the interest rate must increase, leading to a decrease of the capital stock. If $k^* < k_{t-1} < k_t < k_D$ a similar reasoning shows that the capital stock must increase away from k^*. Thus for a positive transfer economy ($Z(k^*, k^*) > 0$) the requirement that the more numerous cohort of young agents balance the positions of the old leads the Golden rule to be unstable.

The intermediary equilibrium that we have presented in this section is not the only way in which an equilibrium can be obtained for which the capital stock trajectory is given by equation (E$'$): if there is a government which runs a deficit or a surplus without adding to it, or if there are assets in the private sector in which agents can invest which pay no dividends (bubbles), or if money (with no explicitly modeled transactions services) can be transmitted from one generation to the next, then the same equation (E$'$) summarizes the equilibrium. These cases have been extensively studied in the economic literature (see e.g Tirole (1985), Weil (1987), Blanchard-Fischer (1989), Azariadis (1993)). However the government debt, or bubble, or money interpretations have never led emphasis to be placed on the property of convergence to the Golden Rule in an economy with underaccumulation: in order that the equilibrium capital stock converges to the Golden Rule rather than the Diamond steady state, either the government has to run a surplus *at all times*, or agents have to invest in negative bubbles, or they must carry "negative" money—all of which are highly implausible situations for the models in question.

On the other hand the possibility that long-lived intermediaries permit intergenerational transfers, although nowhere discussed except in Gale's paper (1973), seems to be a highly likely situation. A possible objection to the concept of an intermediary equilibrium is that it is impossible to initialize because there can never be an old generation which lends to the young, without the agents of this generation having had access to loans when they were young. This argument, while at first sight compelling, underestimates the ability of intermediaries to create loanable funds. Banks began as depositary institutions for wealthy agents and exploited the stochastic nature of the deposit-withdrawal process to accumulate balances which they used to make loans to businesses. In the same way other financial institutions such as insurance companies are able to create loanable funds out of the normal process by which they receive premia in advance of having to reimburse random claims. Thus it seems both natural and realistic to assume that financial intermediaries can indeed create a situation such that firms reimburse more than is needed to reimburse agents who have deposited their savings with them. Note furthermore that the size of the initial surplus needed to initialize the process can be very small, and that once the process has got underway, it is self sustaining.

Finally in the next section, we show that the bond market is not the only financial market that can generate the intergenerational transfers which lead to more investment than in the Diamond equilibrium. The stock market can play the same role provided that the financial value of firms on the stock market can differ from the replacement value of their capital.

The Stock Market and Intergenerational Transfers

Let us return to the Diamond model in which the bond market is restricted to intragenerational loans: thus the purchase of firms by the young and the new investment they undertake must be financed out of their savings. However, instead of assuming that capital is a homogeneous and malleable good, let us assume that, once installed, capital is firm specific and cannot either be transformed back into the consumption good or be transferred from one firm to another without incurring significant “adaptation costs” which, for simplicity, we take to be infinite. Note that the assumption of firm specificity of capital does not introduce any new imperfection into the model, as long as gross investment is positive at every date: since we assume that all firms are equally productive and that new investment is needed at every date, there is no reason to incur the adaptation costs involved in “unbolting” the capital installed a firm. However the nature of firms and the nature of the markets on which they are transferred from one generation to the next must now change. Firms need to become infinitely-lived *corporations* whose ownership is transferred through the sale of equity shares on the stock market.

We thus consider a model in which, in addition to the spot markets for current output and labor services, there are two financial markets—the equity and the bond market. As before, the current output of firms can be channeled either into consumption or used for new investment. However, while ex-ante one unit of investment in any firm costs one unit of current output, ex-post, once the newly invested capital is installed, it is a sunk cost. Let $(1, w_t, (Q_t^j)_{j=1}^J, r_{t+1})$ denote the two spot prices, and the prices and interest rate associated with the firms' equity and the bond at date t. Note that since the firms are infinitely lived, there is the same number J of firms at each date.

The representative agent of the young generation at date t purchases a portfolio of securities

$$(z_t, \theta_t^1, \ldots, \theta_t^J)$$

consisting of an amount z_t of the bond and a share θ_t^j of firm j (for $j = 1, \ldots, J$) so as to maximize lifetime utility $u(c_{0,t}, c_{1,t+1})$, subject to the two budget constraints

$$\begin{aligned} c_{0,t} &= w_t - \sum_{j=1}^{J} \theta_t^j Q_t^j - z_t \\ c_{1,t+1} &= \sum_{j=1}^{J} \theta_t^j (D_{t+1}^j + Q_{t+1}^j) + z_t(1 + r_t) \end{aligned} \tag{17}$$

where D_t^j is the dividend paid by firm j at date t. The agent takes the prices $(w_t, (Q_t^j)_{j=1}^J, r_{t+1})$ as given and correctly anticipates the next period dividends and equity prices $(D_{t+1}^j, Q_{t+1}^j)_{j=1}^J$. The maximum problem of the agent has a solution if and only if the no-arbitrage condition between the stock and bond market

$$Q_t^j = \frac{1}{1 + r_{t+1}}(D_{t+1}^j + Q_{t+1}^j), \quad j = 1, \ldots, J \tag{18}$$

holds for the equity price of each firm. Since by (18) the rate of return on the bond and each of the equity contracts is the same, the agent is indifferent between investing in any firm or investing in the bond market: all that matters is the total sum invested in the capital markets, namely the agent's total savings s_t. When (18) holds the budget equations (17) can be written as

$$\begin{aligned} c_{0,t} &= w_t - s_t \\ c_{1,t+1} &= s_t(1 + r_{t+1}) \end{aligned}$$

where

$$s_t = \sum_{j=1}^{J} \theta_t^j Q_t^j + z_t \tag{19}$$

Thus as in the previous version of the model the agent's optimal choice is characterized by the savings function $s(r_{t+1}, w_t)$. The precise decomposition of the savings between stocks and bond will be determined by the equilibrium conditions.

Consider the investment decision of firm j. To maintain the symmetry assumption among agents, suppose that every young agent has an ownership share in firm j, and that the young agents assemble at a board meeting to decide on firm j's investment plan for date t, the investment being financed by borrowing. The present value of firm j's profit is

$$\frac{1}{1 + r_{t+1}}\Big(F(1-\beta)K_t^j + I_t^j, L_{t+1}^j) - w_{t+1}L_{t+1}^j - (1 + r_{t+1})I_t^j + Q_{t+1}^j\Big)$$

Assuming free disposal of firms, i.e. assuming that the old generation can destroy the capital $(1-\beta)K_{t+1}^j$ if they cannot sell it satisfactorily, Q_{t+1}^j must be non-negative. On the other hand Q_t^j cannot exceed $(1-\beta)K_t^j$ since otherwise the young agents would be better off creating a new firm, purchasing the requisite capital $(1-\beta)K_t^j$ on the current output market. Thus at each date we must have $0 \leq Q_t^j \leq (1-\beta)K_t^j$. We will show how to find an equilibrium of the model assuming that these inequalities are not binding: this will imply that the analysis must be restricted to an appropriate subset of parameters and initial conditions, as we shall see below.

The optimal hiring decision with respect to labor, L_{t+1}^j, satisfies (5) as before, and the optimal choice of investment plan I_t^j satisfies

$$F_K'(K_{t+1}^j, L_{t+1}^j) - (1 + r_{t+1}) + \frac{\partial Q_{t+1}^j}{\partial I_t^j} = 0 \tag{20}$$

In order to choose I_t^j optimally, the young agents must thus anticipate Q_{t+1}^j as a function of I_t^j. We assume that they correctly anticipate the path of future interest rates and wages and that the next generation (amd all subsequent generations) will take optimal decisions once they have bought the firm from them. They thus anticipate that

$$Q^j_{t+1} = \frac{1}{1+r_{t+2}} \Big(F(1-\beta)^2 K^j_t + (1-\beta) I^j_t + I^j_{t+1}, L^j_{t+2}) - w_{t+2} L^j_{t+1} - (1+r_{t+2}) I^j_{t+1} + Q^j_{t+2} \Big) \tag{21}$$

By successive substitutions, Q^j_{t+1} is the infinite sum of dividends given by firm j from date $t+2$ on, plus possibly a bubble component $\lim_{t\to\infty} Q^j_t$. Thus (20) can also be written

$$F'_K(K^j_{t+1}, L^j_{t+1}) - (1+r_{t+1}) + \frac{1}{1+r_{t+2}} \left((1-\beta) F'_K(K^j_{t+2}, L^j_{t+2}) + \frac{\partial Q^j_{t+2}}{\partial I^j_t} \right) = 0 \tag{22}$$

Note that from date $t+1$ on, when $I^j_{t+1} > 0$ one unit of investment at date t is perfect substitute for $(1-\beta)$ of investment at date $t+1$. Thus restricting attention to trajectories on which investment in firm j is positive at all date, by optimality of I^j_{t+1} (using (20) at date $t+1$)

$$\frac{\partial Q^j_{t+2}}{\partial I^j_t} = (1-\beta) \frac{\partial Q^j_{t+2}}{\partial I^j_{t+1}} = (1-\beta) \Big((1+r_{t+2}) - F'_K(K^j_{t+2}, L^j_{t+2}) \Big)$$

which, substituted into (22) gives the same first-order condition (6) as in the Diamond model.

Thus as in the Diamond model, all firms choose the same capital-labor ratio and equations (8, 9,10) must hold for all t in equilibrium to ensure that firms optimize and that the labor market clears. Consider the financial markets: in order that equilibrium can exist (18) must hold for each firm, or equivalently, (21) must hold at each date. In aggregate per capita terms ($q_t = (\sum_{j=1}^J Q^j_t)/N_t$)

$$q_t = \frac{1+n}{1+r_{t+1}} \left(f(k_{t+1}) - w_{t+1} - (1+r_{t+1}) \frac{i_t}{1+n} + q_{t+1} \right) \tag{23}$$

must hold. Then equilibrium on the stock market requires $\theta_t = 1/N_t$ and equilibrium on the bond market requires $z_t = i_t$. Since agents are indifferent between bonds and stocks, these conditions will hold if

$$s(r_{t+1}, w_t) = q_t + i_t \tag{24}$$

Combining (8), (10), (23) and (24) leads to the same second-order difference equation (E$'$) as in the model with intermediaries, and thus to the same dynamics. However some restrictions on the characteristics of the economy are needed to ensure that the inequalities $0 \le Q^j_t \le (1-\beta) K^j_t$ and $I^j_t > 0$ are satisfied along the trajectories. To undestand the restrictions involved, note first that from the analysis of an intermediary equilibrium we know that $Z(k_{t+1}, k_t)$ has the same sign for all t as $Z(k_1, k_0)$. Since

$$Z(k_{t+1}, k_t) = s(r(k_{t+1}), w(k_t)) - (1+n) k_{t+1} = s(r(k_{t+1}), w(k_t)) - (1-\beta) k_t - i_t$$

and since in a stock market equilibrium $s(r(k_{t+1}), w(k_t)) = q_t + i_t$, it follows that

$$Z(k_{t+1}, k_t) = q_t - (1-\beta)k_t$$

Thus if the initial conditions are such that $Z(k_1, k_0) \leq 0$ (i.e $q_0 \leq (1-\beta)k_0$) then at all subsequent dates $q_t \leq (1-\beta)k_t$. If $Z(k_1, k_0) = 0$ then the stock market price of each firm is equal to the replacement value of the capital which is transferred and the equilibrium is the Diamond equilibrium. If $Z(k_1, k_0) < 0$ then $q_t < (1-\beta)k_t$ at every date: the discounts at each period on the equity price of each firm are the transfers from the old (who own the firms) to the young (who buy them), and it is these discount-induced transfers which lead to the Golden Rule level of capital k^* rather than the Diamond capital stock k_D, in economies with underaccumulation.

However, in order that the investment of firms be positive at every date as we have assumed, the sequence of discounts on firms' prices must not become too large. A precise analysis of the firms' optimal investment decisions can be found in Magill-Quinzii (1999). It can be summarized as follows: in order that the young agents who have bought a firm decide to undertake new investment, they must anticipate that they will recover the (depreciated) cost of the investment when they sell the firm next period—this is the property which lies behind the first-order condition $f'(k_{t+1} = r_{t+1} + (1-\beta)$. In order that these expectations are fulfilled it must be the case that $(1+n)q_{t+1} \geq (1-\beta)i_t$ at every period. Thus in order that trajectories with positive investment exist and converge to the Golden Rule, this inequality must be satisfied at k^*. Recalling that $q^* = s(r(k^*), w(k^*)) - i^*$, this requires that

$$s(r(k^*), w(k^*)) \geq i^* + \frac{(1-\beta)}{1+n} i^*$$

Thus we must restrict attention to economies for which the savings of the representative young consumer when faced with the interest rate and wage rate $(r(k^*), w(k^*))$ is sufficient to pay for the current (per-capita) investment i^* and the depreciated investment of the period before $\frac{(1-\beta)}{1+n} i^*$. Thus while the stock market can achieve intergenerational transfers, it is more limited than a bond market with infinite-lived intermediaries in the amount of transfers that it can achieve.

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